MATH-SHU 362

Recitation 2: First Order ODE

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Exercise 1. *Solve the given differential equations*

1. $y' = \frac{t^2}{y}$,
2. $y' + y^2 \sin(t) = 0$.

Exercise 2. *Solve the initial value problem and determine how the interval in which the solution exists depends on the initial value y_0*

1. $y' = -4\frac{t}{y}, y(0) = y_0$,
2. $y' = 2ty^2, y(0) = y_0$.

Exercise 3. *Find some counter-example of first order ODE such that*

1. *the solution is not global;*
2. *the solution is not unique.*

Exercise 4 (Grönwall inequality)**.** *In the following, let α, β are continuous function on I and $t_0, t \in I$ with $t_0 < t$.*

1. *If u satisfies that*
$$u'(t) \leqslant \beta(t)u(t) + \alpha(t), \quad t \in I,$$
then we have
$$u(t) \leqslant \exp\left(\int_{t_0}^{t} \beta(s)\,\mathrm{d}s\right) u(t_0) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} \beta(r)\,\mathrm{d}r\right) \alpha(s)\,\mathrm{d}s, \quad t \in I. \tag{1}$$
2. *If β is non-negative and if u is integrable and satisfies the integral inequality that*
$$u(t) \leqslant \alpha(t) + \int_{t_0}^{t} \beta(s)u(s)\,\mathrm{d}s, \quad t \in I,$$
then we have the estimate that
$$u(t) \leqslant \alpha(t) + \int_{t_0}^{t} \alpha(s)\beta(s) \exp\left(\int_{s}^{t} \beta(r)\,\mathrm{d}r\right) \mathrm{d}s, \quad t \in I. \tag{2}$$
3. *Application: use this result to prove the stability with respect to the initial data. Let u_1, u_2 be two solutions for the system*
$$i \in \{1,2\}, \qquad \begin{cases} \frac{d}{dt}u_i(t) &= f(t, u_i(t)), \\ u_i(0) &= w_i, \end{cases}$$
with f a continuous function satisfies Cauchy-Lipschitz condition
$$\forall t \in \mathbb{R}_+, x, y \in \mathbb{R}, \qquad |f(t,x) - f(t,y)| \leqslant L|x-y|.$$
We fix interval $[0,T]$. Then for every $\varepsilon > 0$, there exists $\delta(\varepsilon, T, L) > 0$ such that once $|w_1 - w_2| \leqslant \delta$, we have $\sup_{t\in[0,T]} |u_1(t) - u_2(t)| \leqslant \varepsilon$.